Exclusion Statistics in Classical Mechanics

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(April 15, 2000)

We present a general method to derive the classical mechanics of a system of identical particles in a way that retains information about quantum statistics. The resulting statistical mechanics can be interpreted as a classical version of Haldane's exclusion statistics.

PACS numbers: 05.30.Pr, 71.10.Pm, 73.40.Hm

Particle statistics enters quantum physics in two related but logically distinct way. The first one is related to the symmetry of the wave function, or more generally to phase factors associated with the exchange of identical particles. The other is related to entropy, *i.e.* to the counting of quantum states, and is expressed through the Pauli exclusion principle and the phenomenon of Bose condensation. Some years ago, Haldane pointed out that it is possible to have a certain kind of quantum statistics of the second type, so called exclusion statistics, without any reference to wave functions or exchange factors [1]. While the exchange phase factors characterizing fermions, bosons or (in 2 space dimensions) anyons [2,3] are intimately connected to quantum mechanics, there is no logical reason for not having effects of the second type of statistics even in classical systems. In fact, as is well known such effects must be put in by hand in order to avoid the Gibbs paradox in classical statistical mechanics.

In this letter we show that it is possible to formulate a classical mechanics that builds in the effects of quantum statistics at the Lagrangian level. The dynamics, *i.e.* the equations of motion, will not depend on the statistics parameter, but the counting of states, and thus the statistical mechanics, will. In the classical description the statistics is expressed through the occupation of phase space volume. Thus, each new particle introduced in the system will reduce the available phase space volume for the other particles, and the degree of reduction defines the classical statistics parameter.

We first sketch a general formulation of the problem, and then present some specific results for two cases, noninteracting charged particles in a strong magnetic field and vortices in the Landau-Ginzburg-Chern-Simons theory. Both these systems are of interest for the quantum Hall effect. In both examples, we show that the resulting classical statistical mechanics is a classical version of Haldane's exclusion statistics. Our examples are two-dimensional, but just as in the case of exclusion statistics, there is no reason in principle for our construction not to work in an arbitrary dimension. Below we will only present the general ideas and some of the main results. A fuller account including calculational details can be found in ref. [4].

Consider a general quantum system and a subset of states $|\psi_{\mathbf{z}}\rangle$, which is labelled by a set of complex coordinates $\mathbf{z} = \{z_1, z_2, ..., z_N\}$. These may be the coordinates of a system of (identical) particles or the coordinates of an N soliton configuration. We only assume that the wave function evolves smoothly with a change of these coordinates, and that it is symmetric under an interchange of any pair of the N coordinates. To define the corresponding classical mechanics, consider the constrained system where the evolution of the full quantum system is projected to the manifold, $\mathcal{M}$, defined by the (normalized) states $|\psi_{\mathbf{z}}\rangle$. The Schrödinger equation of the full system can be derived from the Lagrangian,

$$L = i\hbar\langle\psi|\dot{\psi}\rangle - \langle\psi|H|\psi\rangle \; , \tag{1}$$

and the Lagrangian of the constrained system is obtained from this by restricting $|\psi\rangle$ to the subset of states $|\psi_{\mathbf{z}}\rangle$. Expressed in terms of the coordinates $\mathbf{z}$, it is of the generic form,

$$L(\mathbf{z}, \overline{\mathbf{z}}) = A_{\overline{z}_i}\dot{\overline{z}}_i + A_{z_i}\dot{z}_i - V(z_i, \overline{z}_i) \; , \tag{2}$$

where A_z is the Berry connection,

$$A_{z_i} = i\hbar\langle\psi_z|\partial_{z_i}\psi_x\rangle \; , \tag{3}$$

V is the expectation value of the Hamiltonian in the state $|\psi_{\mathbf{z}}\rangle$.

An important special case is when the state vectors which define $\mathcal{M}$ are, up to normalization, analytic functions of z_i,

$$|\psi_{\mathbf{z}}\rangle = \mathcal{N}(\overline{\mathbf{z}}, \mathbf{z})|\phi_{\mathbf{z}}\rangle \; , \tag{4}$$

where $|\phi_{\mathbf{z}}\rangle$ denotes the analytic part of the state vector, and $\mathcal{N}(\overline{z}, z)$ is the normalization factor. The vector potentials are then given by

$$A_{z_i} = -i\hbar\partial_{z_i}\ln\overline{\mathcal{N}}(\overline{\mathbf{z}}, \mathbf{z}) \; , \tag{5}$$

and $\mathcal{M}$ is a Kähler manifold where the Kähler potential is related in a simple way to the normalization factor,

$$K(\overline{\mathbf{z}}, \mathbf{z}) = \hbar\ln|\mathcal{N}(\overline{\mathbf{z}}, \mathbf{z})|^{-2} \; . \tag{6}$$

The geometry of $\mathcal{M}$ is described by the field strength

$$f_{\overline{z}_i z_j} = \partial_{\overline{z}_i} A_j - \partial_{z_j} A_{\overline{i}} = i\partial_{\overline{z}_i} \partial_{z_j} K(z, \overline{z}) \ , \tag{7}$$

in terms of which the symplectic form, ω, and the metric, ds^2, takes the forms $\omega = -f_{\overline{z}_i z_j} d\overline{z}_i \wedge dz_j$ and $ds^2 = -2if_{\overline{z}_i z_j} d\overline{z}_i dz_j$. The importance of the symplectic form ω is that it determines the Poisson brackets, and thus, together with the Hamiltonian, defines the classical mechanics [5]. The Poisson bracket is

$$\{A, B\} = f^{-1}_{z_i \overline{z}_j} \, \partial_{z_i} A \, \partial_{\overline{z}_j} B \ , \tag{8}$$

and the equation of motion can then be written as

$$\dot{z}_i = \{z_i, V\} \ . \tag{9}$$

To be more specific we now consider a system of charged particles moving in two dimensions in the presence of a strong magnetic field that restricts the available states to the lowest Landau level. In this example we can explicitly derive the metric and symplectic form, and show that they can be obtained from a Kähler potential. For calculations, it is convenient to consider particles moving on a sphere. On a unit sphere penetrated by $2j$ units of magnetic flux a particle with unit charge has a total angular momentum $J = j + L$, where L is the orbital angular momentum, and the lowest Landau level corresponds to $L = 0$ with a $2j + 1$ degeneracy [6]. We shall use the notation of ref. [7] and define a coherent state by rotations of a minimum uncertainty reference state $|0\rangle$,

$$|z\rangle = D(z)|0\rangle = e^{zJ_+} e^{\eta J_0} e^{-\overline{z}J_-} |0\rangle \ . \tag{10}$$

Here the complex coordinate z is defined via a stereographic projection, and the rotation operators, $D(z)$, form a unitary and irreducible representation of the rotation group, generated by J_m, $\eta = \ln(1 + \overline{z}z)$, and $|0\rangle$ is annihilated by J_+.

Fully symmetrized and antisymmetrized states corresponding to fermions and bosons are given by

$$|\mathbf{z}, \pm\rangle = \mathcal{N}(\mathbf{z}, \overline{\mathbf{z}}) \frac{1}{\sqrt{N!}} \sum_P \eta_P^{\pm} e^{z_{i_P} J_+^i} |\mathbf{0}\rangle \ , \tag{11}$$

with $\eta_P^{\pm}$ the appropriate sign for the permutation P. The normalizations of these states are readily obtained from the properties of the $D(z)$:s,

$$|\mathcal{N}(\mathbf{z}, \overline{\mathbf{z}})|^{-2} = \sum_P \eta_P \prod_i \left(1 + \overline{z}_{i_P} z_i\right)^{2j} \ . \tag{12}$$

For the case of N coinciding bosons, $z_i = z$, we immediately get the following Kähler potential

$$K(z, \overline{z}) = N\hbar 2j \ln(1 + \overline{z}z) \ , \tag{13}$$

and the corresponding metric, $ds^2 = 2N\hbar/(1 + \overline{z}z)^2 dz d\overline{z}$, is just N times that of a sphere.

To assess the effect of statistics in the classical description we calculate the N-particle phase space volume. Following Manton [8], and Samols [9] we can use (13) to obtain the N-particle volume from the volume of N coinciding bosons, and the result is,

$$V_B = \frac{1}{N!} (A)^N \ , \tag{14}$$

with A as the volume of the single-particle space,

$$A = h \int_{sph} \omega = h2j = \frac{\hbar 4\pi R^2}{l^2} = e\Phi \ , \tag{15}$$

which is h times the number of flux quanta $\phi_0 = h/e$ that penetrate the sphere. Thus, for bosons the only effect of the indistinguishability of the particles is the factor $1/N!$, and there is no further reduction in phase space volume.

One should note that the classical phase space defined as above is everywhere a smooth manifold. This is different from the case when the N-particle space is defined as a product of single particle spaces with identification of equivalent configurations. In the latter case the points corresponding to a coincidence of two particle positions are singular. Also note that the factor $1/N!$ here appears naturally from the geometry, not through any additional assumption about identification of points.

For fermions a similar but technically more involved calculation can be done. The resulting phase space volume is,

$$V_F = \frac{1}{N!} \left(A - (N-1)h\right)^N \ . \tag{16}$$

Compared with the Bose case there is an additional reduction of phase space. The available phase space for any particular fermion is reduced with an amount h by each of the other particles present in the system. This can be understood as a *classical version of the Pauli exclusion principle,* and is consistent with the usual semi-classical interpretation of quantum mechanics, where each quantum state is associated with a phase space volume h. Note that there is a maximum number of particles allowed, $N = 2j + 1$, in which case the phase space volume (16) vanishes. This corresponds to all states with zero orbital angular momentum being filled, *i.e.* to a filled lowest Landau level, which is an incompressible state.

The calculation can be done also in the case of anyons although there is no explicit expression for the normalization constant corresponding to (12). For details we again refer to [4] and only quote the result,

$$V_\nu = \frac{1}{N!} \left(A - \nu(N-1)h\right)^N \ , \tag{17}$$

where the exchange phase of the anyons is $\nu\pi$.

The expressions we have found above for the N-particle phase space volume demonstrates a *classical fractional*

exclusion principle. Thus, each new particle introduced in the system will find the available volume reduced by $\alpha = \nu h$ relative to the previous one. The quantity α, *i.e.* the reduction in phase space volume, can be taken as defining the classical statistics parameter of the particles. In the present case it is simply the (dimensionless) quantum statistics parameter ν multiplied with Planck's constant h.

We next consider a classical field theory with soliton solutions, namely the Chern-Simons Ginzburg-Landau (CSGL) theory, originally introduced as a field theory for the quantum Hall effect [10],

$$L = \int d^2x [i\hbar\phi^* D_0\phi - \frac{\hbar^2}{2m}|\vec{D}\phi|^2 - \frac{\lambda}{4}(|\phi|^2 - \rho_0)^2 + \mu\hbar\epsilon^{\mu\nu\rho} a_\mu \partial_\nu a_\rho] , \quad (18)$$

where ϕ is a complex matter field, a_μ a Chern-Simons field, m a mass parameter, λ the interaction strength, ρ_0 the preferred density of the system and μ a statistics parameter. For the original Laughlin quantum Hall states described by the model the statistics parameter takes the values $\mu = 1/[4\pi(2k+1)]$. This theory has vortices (quasi-particles) as soliton solutions. In a certain approximation the dynamics can be described in terms of vortex coordinates alone, and a phase space description can be derived from the full theory. Again it is possible to calculate the phase space volume corresponding to a N-vortex solution.

The vortex configurations can be parameterized by a set of vortex coordinates $\mathbf{z} = \{z_1, z_2, ..., z_n\}$, just as the charged particles in a magnetic field discussed above. The precise form of the multi-vortex configurations for given coordinates is not known, but for critical coupling $(\lambda = 1)$ the existence of N-vortex configurations with arbitrary positions can be deduced [11]. With the system constrained to the manifold of N-vortex configurations, a classical mechanics follows with a kinetic term for the N-vortex system corresponding to a phase space with Kähler metric [4]. By use of earlier results obtained by Manton for the related relativistic abelian Higgs model [8] we find for the phase space volume of N vortices,

$$V_N = \frac{1}{N!}(A - 4\pi\mu h(N-1))^N , \quad (19)$$

where the classical statistics parameter, as determined by the reduction in available phase space due to the presence of other vortices, is $\alpha = 4\pi\mu h \equiv gh$. We can interpret g, the classical parameter divided by h, as the dimensionless quantum statistics parameter. The value $g = 4\pi\mu$ agrees with the value of the statistics parameter as determined from Berry phase calculations with Laughlin wave functions [12], or from the properties of vortices in the CSGL model [10].

We now discuss the statistical mechanics of the classical systems just derived. Both these systems have the special property that the energy does not depend on the state, but only on the number of particles. This means that the statistical mechanics is determined by the phase space volume V_N, which has been determined in the previous sections, and by the energy E_N. The classical partition function is simply the total number of states, V_N/h^N multiplied with the Boltzmann factor, *i.e.*

$$Z_N = \frac{V_N}{h^N} e^{-\beta E_N} , \quad (20)$$

Using standard thermodynamics we get for the entropy

$$S = N\ln(1-\alpha\rho) + N\ln\frac{A}{h} - N\ln N + N , \quad (21)$$

where $\alpha = \nu h$ or gh, and where we have introduced the classical phase space density $\rho = N/A$ and neglected the difference between N and $N-1$, which is irrelevant in the thermodynamic limit.

In the systems we have considered the real two-dimensional space where the particles or vortices move is proportional to the phase space. Defining the pressure as, $P = -(\partial F/\partial A)_T$, where $A = V_1$ is the phase space volume for a single particle, we then get, by use of standard thermodynamic relations, the equation of state

$$\beta P = \rho/(1-\alpha\rho). \quad (22)$$

This expression shows that there is a maximum density $\rho = 1/\alpha$ allowed by the system, which corresponds to an infinite pressure and therefore to an incompressible state. For the phase space volume this means $V_N = 0$, *i.e.* there is no available phase space volume for any new particle added to the system. For the anyon system this situation corresponds to a completely filled Landau level. What is unusual about this is that the blocking, which can be interpreted as representing a generalized Pauli principle, shows up not only in the quantum, but also in the classical description of the system.

Finally we show that the thermodynamics just derived can be viewed as a classical limit of Haldane exclusion statistics [1].

The statistical mechanics of particles with exclusion statistics can be derived by assuming that the total energy can be written as a sum of single-particle energies and using the prescriptions for statistical weight given by Haldane separately for each single-particle energy level [13–15]. The result for the entropy is

$$S = \sum_k D_k\{[1+(1-g)n_k]\ln[1+(1-g)n_k] + (1-gn_k)\ln(1-gn_k) - n_k\ln n_k\} , \quad (23)$$

where the sum runs over single-particle energy states, and g is the exclusion statistics parameter. D_k is the degeneracy of the k-th level and the quantum distribution

function n_k is the average occupation number of the state k.

Since each quantum state occupies the phase space volume $h^{\mathcal{D}}$, with $2\mathcal{D}$ the dimension of the single-particle phase space, we can relate n and ρ in the semiclassical limit by $n = \rho h^{\mathcal{D}}$. In the Boltzmann limit, $h \to 0$ and $n \to 0$, all dependence on g in (23) goes away. If we, however, define the classical physics by the limit $h \to 0$, $g \to \infty$ and $gh^{\mathcal{D}} \to \alpha$, where α is interpreted as a classical statistics parameter (23) gets the nontrivial limit of

$$S = \sum_k D_k h^{\mathcal{D}} \left[\rho_k \ln(1 - \alpha\rho_k) - \rho_k \ln(\rho_k h) + \rho_k\right] \ . \quad (24)$$

If we further specialize to the case of fully degenerate states in a two-dimensional phase space, where the sum is simply replaced by the total number of available single-particle states, $G = A/h$, and where ρ_k is replaced by N/A, we exactly regain (21). This demonstrates that the classical statistical mechanics discussed in the previous section can be regarded as a special limit of exclusion statistics, different from the Boltzmann limit. Starting from the the equation of state for exclusion statistics particles with the same energy, we can also derive (22) if we identify the the physical volume $\mathcal{V}$ with the one particle phase space volume A.

One can also obtain exact results when the particles move in an external harmonic potential. Again one finds equivalence between the statistical mechanics derived from the classical mechanics of identical particles with a statistics parameter and the statistical mechanics derived from exclusion statistics in the classical limit discussed above [4].

Although the phase space in the examples discussed in this paper are two-dimensional, there is no obvious reason for the construction not to work in higher dimensions. One point of particular interest to study further is how the "classical fermion" theory discussed here can be applied to systems of interacting fermions moving in two or three dimensions.

We thank A. Karlhede for discussions and comments on the manuscript. T. H. Hansson and U. Lindström were supported by the Swedish Natural Science Research Council. S. Isakov acknowledges the support received through a NATO Science Fellowship granted by the Norwegian Research Council, and also appreciates the warm hospitality of NORDITA during his stay there in the summer of 1999, where part of this work was done.

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